Tilting

Joseph Sullivan

March 2025

Contents

1 Introduction

This set of notes tries to explain a certain correspondence between coherent sheaves and quiver representations.

To understand this story, we'll need to do understand

- endomorphisms of certain nice generator sheaves, called tilting sheaves

- how to compute with quiver representations.

These notes will assume some working familiarity with triangulated categories—the reference [4] has a nice treatment, also see my other notes.

These notes are also not really intended to be read from start to finish—skip the quiver section and go back as needed.

2 Quiver Algebras

We're going to need to understand a bit of the representation theory of quivers. This is more or less copied from [1], except we switched the convention of path composition and from right modules to left modules.

2.1 Definitions

Definition 2.1. A **quiver** $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of sets Q_0 called the **vertices**, Q_1 called the **arrows**, and maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its **source** $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$.

We will assume all quivers are finite and connected (the underlying the undirected graph is connected).

Definition 2.2. A **path** in Q is a sequence

$$(a \mid \alpha_1, \alpha_2, \ldots, \alpha_\ell \mid b)$$

where $\alpha_k \in Q_1$ are arrows with $t(\alpha_k) = s(\alpha_{k+1})$ (unless $k = 0, \ell$, in which case $a = s(\alpha_1)$ and $t(\alpha_\ell) = b$). We allow $\ell = 0$ when $a = b$ for a trivial path

$$\varepsilon_a = (a \| a).$$

Definition 2.3. Let k be a field and Q a quiver, the **quiver algebra** (or **path algebra**) kQ is the K-algebra with underlying vector space

$$kQ = \bigoplus k \cdot (a \mid \alpha_1, \ldots, \alpha_\ell | b)$$

the free k-space on the set of paths in Q, and multiplication defined on the generators by

$$(c \mid \beta_1, \ldots, \beta_k \mid d) \cdot (a \mid \alpha_1, \ldots, \alpha_\ell \mid b) = \begin{cases} (a \mid \alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_k \mid d) & b = d \\ 0 & \text{otherwise,} \end{cases}$$

(i.e., composition of paths), and the multiplication is extended to all of kQ by distributivity.

Denote kQ_m the k-vector subspace of kQ spanned by paths of length m. This makes kQ into a graded k-algebra.

Remark 2.4. The associative k-algebra kQ is unital. We claim that $\sum_{a\in Q_0}\varepsilon_a$ is a unit for multiplication, since for a given path γ, it will end at some $t(\gamma)$, and then

$$\varepsilon_a\cdot\gamma=\begin{cases}\gamma & a=t(\gamma)\\ 0 & \text{otherwise,}\end{cases}$$

so indeed $(\sum\varepsilon_a)\cdot\gamma=\gamma$. The same argument shows $\gamma\cdot(\sum\varepsilon_a)=\gamma$, so we have

$$\sum_{a\in Q_0}\varepsilon_a=1.$$

Definition 2.5. Let Q be a quiver and kQ its quiver algebra. Define the **arrow ideal** of KQ to be $R_Q:=\langle Q_1\rangle_{kQ}$ the (two sided) ideal generated by the arrows (paths of length 1). In this case,

$$R_Q=\bigoplus_{m\geq 1}kQ_m,$$

and more generally

$$R_Q^\ell=\bigoplus_{\ell\geq m}kQ_\ell.$$

Definition 2.6. A two-sided ideal $I\leq kQ$ is called **admissible** if there exists $m\geq 2$ such that

$$R_Q^m\subseteq I\subseteq R_Q^2,$$

i.e., I is generated by formal sums of paths of length at least 2, and any long enough path is in I.

We call a pair (Q,I) a **bound quiver**, and the k-algebra kQ/I the **bound quiver algebra**.

Every admissible ideal turns out to be generated by finitely many **relations**, i.e., a k-linear sum of paths (of length ≥ 2) with the same source and target.

Definition 2.7. Let (Q,I) be a bound quiver. A **bound quiver representation** is an assignment (M_a,φ_α) of k-vector spaces M_a for $a\in Q_0$ and k-linear maps $\varphi_\alpha:M_{s(\alpha)}\to M_{t(\alpha)}$ for $\alpha\in Q_1$ where for each relation $\sum\lambda_i\gamma_i\in I$,

$$\sum\lambda_i\varphi_{\gamma_i}=0,$$

where for a path $\gamma=(a\mid\alpha_1\cdots\alpha_\ell\mid b)$,

$$\varphi_\gamma=\varphi_{\alpha_\ell}\circ\cdots\circ\varphi_{\alpha_1}.$$

A morphism of bound quiver representations $(M_a,\varphi_\alpha)\to(N_a,\psi_\alpha)$ consists of morphisms of vector spaces $M_a\to N_a$ commuting with $\varphi_\alpha,\psi_\alpha$. This makes bound quiver representations into an abelian category, denoted

$$\mathrm{Rep}_k(Q,I).$$

When all vector spaces are finite dimensional, call the representation a **finite dimensional** representation, and the full subcategory of these is denoted

$$\mathrm{rep}_k(Q, I),$$

which is also abelian.

Construction 2.8. Let (Q, I) be a bound quiver, and (M_a, φ_α) be a bound quiver representation. We get a left (kQ/I)-module M as follows. Define the underlying k-vector space by

$$M := \bigoplus_{a \in Q_0} M_a,$$

and define the kQ-action by first defining for $\alpha \in Q_1$

$$\alpha \cdot \sum_{a \in Q_0} m_a := \varphi_\alpha \cdot m_{s(\alpha)},$$

and then extending by composition/linearity. One can check that I is in the annihilator of M, so this descends to a kQ/I action on M.

Conversely, let M be a left (kQ/I)-module. Define a bound quiver representation by setting for $a \in Q_0$

$$M_a = \varepsilon_a M$$

and for $\alpha \in Q_1$ with $a = s(\alpha)$, $b = t(\alpha)$, denote $L_\alpha : M \to M$ left multiplication by α, and set

$$\varphi_\alpha = L_\alpha|_{M_a} : M_a \to M_b$$

(where $\alpha \cdot M_a = \alpha \cdot \varepsilon_a M = \alpha \cdot M$ lands in $M_b = \varepsilon_b M$ because $\alpha \cdot M = \varepsilon_b \alpha \cdot M \subseteq \varepsilon_b M$). One can check that I in the annihilator of the kQ-action on M will make (M_a, φ_α) satisfy the relations of I.

One can show that this gives an equivalence of categories

$$\mathrm{Rep}_k(Q, I) \simeq A\text{-Mod},$$

which restricts to an equivalence

$$\mathrm{rep}_k(Q, I) \simeq A\text{-mod}.$$

2.2 Projective Resolutions

To make some explicit calculations of left derived functors out of $D^b(\mathrm{rep}_k(Q, I))$, we will need to learn how to build projective resolutions of quiver representations.

We observe that as left kQ/I modules,

$$kQ/I = \bigoplus_{a \in Q_0} (kQ/I)\varepsilon_a,$$

i.e., every path starts at some unique $a \in Q_0$. This shows that each $(kQ/I)\varepsilon_a$ is a projective object, which we will use to start building our projective resolutions.

Construction 2.9. We can start any projective resolution as follows. Let M be a left (kQ/I)-module, then we first surject onto M by

$$\begin{aligned} \bigoplus_{a \in Q_0} (kQ/I)\varepsilon_a \otimes_k \varepsilon_a M &\xrightarrow{\varepsilon} M \\ \varepsilon_a \otimes m_a &\longmapsto m_a, \end{aligned}$$

(and extended by linearity) which is indeed surjective because

$$M = 1 \cdot M = \left(\sum_{a \in Q_0} \varepsilon_a \right) \cdot M = \bigoplus_{a \in Q_0} \varepsilon_a M.$$

Now, we need to generate the kernel. We notice some more "obvious" elements of the kernel, where for any arrow $\alpha \in Q_1$ and $m_{s(\alpha)} \in \varepsilon_{s(\alpha))} M$,

$$\alpha \otimes m_{s(\alpha)} - \varepsilon_{t(\alpha)} \otimes (\alpha \cdot m_{s(\alpha)}) \longmapsto \alpha \cdot m_{s(\alpha)} - \alpha \cdot m_{s(\alpha)} = 0.$$

We can then at least get some of the kernel of ε by

$$\begin{aligned} \bigoplus_{\alpha \in Q_1} (kQ/I)\varepsilon_{t(\alpha)} \otimes_k \varepsilon_{s(\alpha)} M &\xrightarrow{\varphi_1} \bigoplus_{a \in Q_0} (kQ/I)\varepsilon_a \otimes_k \varepsilon_a M \\ \varepsilon_{t(\alpha)} \otimes m_{s(\alpha)} &\longmapsto \alpha \otimes m_{s(\alpha)} - \varepsilon_{t(\alpha)} \otimes (\alpha \cdot m_{s(\alpha)}). \end{aligned} \tag{$*$}$$

We claim that the image of φ_1 is exactly $\ker \varepsilon$, so that we are successful at least with our first step. We will prove this in the following proposition, which gives us an exact sequence

$$\bigoplus_{\alpha \in Q_1} (kQ/I)\varepsilon_{t(\alpha)} \otimes_k \varepsilon_{s(\alpha)} M \xrightarrow{\varphi_1} \bigoplus_{a \in Q_0} (kQ/I)\varepsilon_a \otimes_k \varepsilon_a M \xrightarrow{\varepsilon} M \to 0.$$

When $I = 0$ and Q acyclic, we will see that this is left exact, and we are done with our projective resolution, though in general we will need more steps.

Proposition 2.10. Let Q be acyclic. Then, the image of φ_1 in $(*)$ is the kernel of ε.

Proof. By construction, $\mathrm{im}\varphi_1 \subseteq \ker \varepsilon$, so our task is to show that $\ker \varepsilon / \mathrm{im}\varphi_1 = 0$.

Let $s \in \ker \varepsilon$ be a representative of some element in the quotient. Define the *support* $\mathrm{supp}(s) \subseteq Q_0$ the set of vertices $i \in Q_0$ such that

$$\varepsilon_i s \neq 0.$$

We can then rephrase our task as finding $t \in \ker \varepsilon$ with $s \equiv t \mod \ker \mathrm{im}\varphi_1$ such that $\mathrm{supp}(t) = \emptyset$. We will prove this by induction—but what to induct on? We produce a filtration of our quiver Q as follows. Since $Q^0 := Q$ is finite and acyclic, it contains a *sink*, i.e., a vertex with no outgoing arrows. Removing this vertex, (and taking the full subquiver) we will still have a finite acylic quiver Q^1, and we can repeat, giving us

$$\emptyset = Q^n \leq \cdots \leq Q^1 \leq Q^0 = Q.$$

Now, this filtration gives us an index set to induct on—assuming s is supported on Q^m, we show $s \equiv t$ where t is supported on Q^{m+1}. Denote $\{b\} = Q^{m+1} \setminus Q^m$ the removed sink.

Now, we write down an expression for $\varepsilon_b s$, and subtract elements from $\mathrm{im}\varphi_1 \cap Q^m$ to "cancel it out", to get us our desired $t \in Q^{m+1}$.

We know $\varepsilon_b s$ lives in

$$\varepsilon_b \cdot \bigoplus_{i \in Q_0} (kQ/I)\varepsilon_i \otimes_k \varepsilon_i M,$$

so we can write

$$\varepsilon_b s = \sum_{i \in Q_0^m} \sum_{\substack{\text{paths} \\ \gamma : i \to b}} \lambda_\gamma \gamma \otimes m_\gamma,$$

where $m_\gamma \in \varepsilon_i M$. Separating out the $i = b$ summand (and since ε_b is the only path $b \to b$ by acyclicity),

$$\varepsilon_b s = \varepsilon_b \otimes m_b + \sum_{i \in Q_0^{m+1}} \sum_{\substack{\text{paths} \\ \gamma : i \to b}} \gamma \otimes m_\gamma. \tag{$**$}$$

We are assuming $\varepsilon(s) = 0$, so $\varepsilon_b \varepsilon(s) = \varepsilon(\varepsilon_b s) = 0$, which tells us

$$\varepsilon_b m_b + \sum_{i \in Q_0^{m+1}} \sum_{\substack{\text{paths} \\ \gamma : i \to b}} \gamma m_\gamma = 0. \tag{$***$}$$

Now, using $\mathrm{im}\varphi_1 \subseteq \ker \varepsilon$, we can produce elements

$$\gamma \otimes m_\gamma - \varepsilon_b \otimes \gamma \cdot m_\gamma,$$

which are also in Q^m (by iteratively moving an arrow across from the left argument to the right argument of the tensor). So, subtracting all of these (ranging over all $i \in Q_0^{m+1}$ and all $\gamma : i \to b$) from s, we will get in $(**)$ just a single tensor

$$\varepsilon_b \otimes m_b' = \varepsilon_b \otimes (m_b - \sum \gamma \otimes m_\gamma),$$

so that $(***)$ turns into $m_b' = 0$, which tells us for

$$t := s - \sum_{i \in Q_0^{m+1}} \sum_{\substack{\text{paths} \\ \gamma : i \to b}} (\gamma \otimes m_\gamma - \varepsilon_b \otimes \gamma m_\gamma) \equiv s \mod \mathrm{im}\varphi_1$$

we get as desired $\varepsilon_b t = 0$. So, t is supported on Q^{m+1}, completing our induction. □

Proposition 2.11. Let Q be acyclic. When $I = 0$, the map φ_1 in $(*)$ is injective.

Proof. TODO. □

3 Tilting Equivalence

3.1 Exceptional Collections and Tilting Sheaves

Definition 3.1. Let $\mathcal{D}$ be a k-linear triangulated category.

- An object $E \in \mathcal{D}$ is **exceptional** if it has no nontrivial Exts, i.e.,

$$\mathrm{Hom}_{\mathcal{D}}(E, E[\ell]) = \begin{cases} k \cdot \mathrm{id}_E & \text{if } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

- A sequence of objects

$$E_1, \ldots, E_n$$

is **exceptional** if all objects are exceptional, and there are no "backwards" Exts, i.e.,

$$\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[\ell]) = \begin{cases} k & \text{if } \ell = 0,\ i = j \\ 0 & \text{if } \ell \neq 0,\ i = j \\ 0 & \text{if } i > j. \end{cases}$$

- A sequence is **full** if it generates $\mathcal{D}$ (i.e., any strictly full triangulated subcategory containing the sequence is equal to $\mathcal{D}$).
- A sequence is **strong** if in addition to being exceptional, the remaining (forward) Exts are all concentrated in degree 0, i.e.,

$$\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[\ell]) = \begin{cases} k & \text{if } \ell = 0, i = j \\ 0 & \text{if } \ell \neq 0. \end{cases}$$

(so you are still allowed to have "forwards" Homs).

Example 3.2. Consider the following sequence in $D^b(\mathbb{P}^n)$

$$\mathcal{O}(-n), \mathcal{O}(-n+1), \ldots, \mathcal{O}.$$

Making use of the fact

$$\mathrm{Ext}^\ell(\mathcal{L}, \mathcal{F}) = \mathrm{Ext}^\ell(\mathcal{O}, \mathcal{F} \otimes \mathcal{L}^\vee) = H^\ell(\mathcal{F} \otimes \mathcal{L}^\vee),$$

we can check a few properties of the sequence by standard cohomology computations.

First, the sequence is an exceptional collection, since

$$\mathrm{Ext}^\ell(\mathcal{O}, \mathcal{O}(-i)) = H^\ell(\mathcal{O}(-i)),$$

and now,

- if $\ell = 0$, then we have $H^0(\mathcal{O}(-i))$, which is k if $i = 0$ and 0 otherwise,

- if $\ell = n$, then we have $H^n(\mathcal{O}(-i)) = H^0(\mathcal{O}(i-n-1)) = 0$ for $i \leq n$.
- for any other value of ℓ, we get 0.

Furthermore, this sequence is also strong, since

$$\mathrm{Ext}^\ell(\mathcal{O}, \mathcal{O}(i)) = H^\ell(\mathcal{O}(i)),$$

which is always zero for $\ell \notin \{0, n\}$, and even when $\ell = n$,

$$H^n(\mathcal{O}(i)) = H^0(\mathcal{O}(-i-n-1)) = 0$$

for $i \geq 1$ (or even $i \geq -n$, but we cannot add more twists to our collection without losing the adjective "exceptional").

Beilinson proved that this sequence is full—this is somewhat tricky, relying on a certain "resolution of the diagonal" in $\mathbb{P}^n \times \mathbb{P}^n$ and some Fourier-Mukai machinery.

Remark 3.3. Exceptional objects are particularly simple. For $E \in \mathcal{D}$ be exceptional, we claim that the triangulated category generated by E is

$$\langle E \rangle = \{\bigoplus_i E[i]^{\oplus a_i} \mid \text{all but finitely many } a_i = 0\}.$$

This is clearly closed under direct sums and shifts. To see that this is closed under cones, consider a morphism

$$\bigoplus_i E[i]^{\oplus a_i} \to \bigoplus_i E[i]^{\oplus b_i}.$$

This is given by some matrix in $\mathrm{Hom}(E[i]^{\oplus a_i}, E[j]^{\oplus b_j})$ (as always for morphisms between direct sums), but it is in fact diagonal because E is exceptional, so the cone over this matrix splits as a direct sum of (shifts of) cones over morphisms

$$E^{a_i} \to E^{b_i}.$$

which is just a matrix in k. By Gaussian elimination, we can hit both sides with automorphisms (which only changes the cone up to isomorphism) so that the matrix only has diagonal 1 entries, and so our cone splits as a direct sum of cones over

$$E \xrightarrow{\mathrm{id}_E} E \qquad \text{or} \qquad E \xrightarrow{0} E,$$

so we either get 0 or $E \oplus E[1]$. In any case, the cone will be a direct sum of objects we are including, so indeed $\langle E \rangle$ is given by our description.

This tells us that $\langle E \rangle \cong D^b(k)$, (a priori objects of $D^b(k)$ look like complexes, but because it has homological dimension ≤ 1, the complexes split as direct sums of their cohomologies, which gives us the same form as $\langle E \rangle$).

Full strong exceptional collections are also very simple—they form a particularly nice generating set for the triangulated category. For a single exceptional object, we got one of the easiest k-linear triangulated categories there is, which is $D^b(k)$. Now, for a collection, we'll get something a little more complicated, but still very controllable.

We first pass from exceptional collections to a related object called a *tilting object*, but we need to give another definition related to generating a triangulated category.

Definition 3.4. We say that a full triangulated subcategory $\mathcal{D}'$ of a triangulated category $\mathcal{D}$ is **saturated/épaisse** if it closed under direct summands.

Write $\langle E\rangle$ for the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing E, and say that E **classically generates** $\mathcal{D}$ if $\langle E\rangle = \mathcal{D}$.

Warning 3.5. Sometimes people write $\langle S\rangle$ to mean the extension closure of S, i.e., the smallest strictly full subcategory containing S and closed under *extensions*, rather than cones/shifts/direct summands. Always check what someone means.

Remark 3.6. For any $E \in \mathcal{D}$, there is actually an explicit description of $\langle E\rangle$. To get a strictly full saturated triangulated subcategory containing $\langle E\rangle$, one might start with $\{E\}$, and then repeatedly throw in shifts, cones, direct sums, direct summands, etc. The miracle is that if you do this countable many times (unioning the results), you actually get all of $\langle E\rangle$ [2, Lemma 13.36.2]. This then gives us the following induction strategy to prove properties of $\langle E\rangle$ [2, Remark 13.36.7].

Induction. Let T be a property enjoyed by objects of $\mathcal{D}$ (where we write $T(A)$ to mean T holds for A), and suppose

(1) (base case) $T(E[n])$ for all $n \in \mathbb{Z}$,

(2) if $T(K)$ and $T(L)$, then $T(K \oplus L)$,

(3) if $T(K \oplus L)$ then $T(K)$ and $T(L)$,

(4) if $K \to L \to M \to K[1]$ is a distinguished triangle and T holds for two of them, then T holds for the third,

then T holds for all objects in $\langle E\rangle$.

Definition 3.7. Let X be a smooth projective variety over k. A sheaf $T \in \mathrm{Coh}(X)$ is called a **tilting sheaf** if

(T1) the algebra $A := \mathrm{End}(T)$ has finite global dimension (i.e., there exists $d \in \mathbb{Z}_{\geq 0}$ such that any module admits a projective resolution of length less than d).

(T2) The modules $\mathrm{Ext}^\ell(T, T) = 0$ for all $\ell > 0$.

(T3) The object classically generates $D^b(X)$.

In fact, more is true. An object X of a triangulated category is called *compact* if $\mathrm{Hom}(X, -)$ preserves all coproducts in the triangulated category. In $\mathrm{Coh}(X)$, all coproducts are products, so $\mathrm{Hom}(X, -)$ automatically preserves coproducts, and so a classical generator will actually be a compact generator.

Proposition 3.8. Let $E_1, \ldots, E_r \in \mathcal{A}$ be a full strong exceptional collection for $D^b(\mathcal{A})$. Then,

$$T = \bigoplus_{i=1}^{r} E_i$$

is a tilting object.

Proof. (T2) and (T3) are reasonably quick to prove.

To see (T1), we can see that

$$A = \text{End}(T)$$

is equivalent to the k-algebra

$$\begin{bmatrix} \text{Hom}(E_1, E_1) = k & \text{Hom}(E_1, E_2) & \cdots & \text{Hom}(E_1, E_n) \\ 0 & \text{Hom}(E_2, E_2) = k & \cdots & \text{Hom}(E_2, E_n) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \text{Hom}(E_n, E_n) = k \end{bmatrix}$$

with multiplication given by matrix multiplication/composition of Homs. So, A is equivalent to the quotient algebra $kQ/\langle R \rangle$ for an acyclic bound quiver (Q, R), and the general theory says that these have finite global dimension. □

Remark 3.9. Regarding the algebra structure of A, we make some comments. The Jacobson radical J of A will be the strictly upper triangular matrices in our suggestive matrix. To see this, set J the direct sum of all Homs which are not $\text{Hom}(E_i, E_i)$ (the strictly upper triangular matrices), and check (via [1, p. I.1.4]) that J is nilpotent and A/J is a product of copies of $\mathbb{C}$.

First, J is nilpotent because J^ℓ only consists of formal sums of morphisms $E_i \to E_{i+\ell}$, so when $\ell \geq r$, there are no more morphisms to sum over.

Next, A/J is isomorphic to

$$\prod_{i=1}^{r} \text{Hom}(E_i, E_i) = \prod_{i=-n}^{0} \mathbb{C},$$

so indeed a product of fields.

The Jacobson radical is really the tool to find the quiver Q. We set the vertices of Q to be a chosen basis for A/J, so conveniently we'll choose the basis to be $\text{id}_{E_1}, \ldots, \text{id}_{E_r}$. Next, we set the arrows from E_i to E_j to be a chosen basis for $\text{id}_{E_j}(J/J^2)\text{id}_{E_i}$. We give a summary of the theory of quiver representations in section 2.

Example 3.10. Let's understand for

$$A = \text{End}(\bigoplus_{i=-n}^{0} \mathcal{O}(i))$$

how to see A as a bound quiver algebra. Our suggestive matrix is the following.

$$\begin{bmatrix} k\cdot \mathrm{id}_{\mathcal{O}(\text{-}n)} & \mathrm{Hom}(\mathcal{O}(\text{-}n),\mathcal{O}(\text{-}n+1)) & \cdots & \mathrm{Hom}(\mathcal{O}(\text{-}n),\mathcal{O}) \\ 0 & k\cdot \mathrm{id}_{\mathcal{O}(\text{-}n+1)} & \cdots & \mathrm{Hom}(\mathcal{O}(\text{-}n+1),\mathcal{O}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k\cdot \mathrm{id}_{\mathcal{O}} \end{bmatrix} = \begin{bmatrix} k & k^{\binom{n+1}{1}} & \cdots & k^{\binom{n+n}{n}} \\ 0 & k & \cdots & k^{\binom{n+n-1}{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{bmatrix}$$

As explained earlier, J is the upper triangular matrices. To compute the arrows from $\mathcal{O}(i)$ to $\mathcal{O}(j)$, we compute

$$\mathrm{id}_{\mathcal{O}(j)}(J/J^2)\mathrm{id}_{\mathcal{O}(i)} = \mathrm{id}_{\mathcal{O}(j)}J\mathrm{id}_{\mathcal{O}(i)}/\mathrm{id}_{\mathcal{O}(j)}J^2\mathrm{id}_{\mathcal{O}(i)}.$$

As a first step, we compute $\mathrm{id}_{\mathcal{O}(j)}J\mathrm{id}_{\mathcal{O}(i)}$, which is exactly $\mathrm{Hom}(\mathcal{O}(i),\mathcal{O}(j))$. Next, we compute

$$\mathrm{id}_{\mathcal{O}(j)}J^2\mathrm{id}_{\mathcal{O}(i)},$$

which will be the subset of $\mathrm{Hom}(\mathcal{O}(i),\mathcal{O}(j))$ that factor as a sum of maps which go through some strictly intermediate $\mathcal{O}(\ell)$. If $j = i+1$, then no strict factorization is possible. If $j > i+1$, then we always have a strict factorization, since a map $\mathcal{O}(i) \to \mathcal{O}(j)$ will be given by a degree $j-i$ homogeneous polynomial, which we can write as a sum of monomials, and then peeling one monomial away we get the factorization.

So,

$$\mathrm{id}_{\mathcal{O}(j)}(J/J^2)\mathrm{id}_{\mathcal{O}(i)} \cong \begin{cases} \mathrm{Hom}(\mathcal{O}(i),\mathcal{O}(i+1)) & j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we should place $n+1$ arrows between $\mathcal{O}(i)$ and $\mathcal{O}(i+1)$ for a chosen basis

$$\mathrm{Hom}(\mathcal{O}(i),\mathcal{O}(i+1)) = \langle x_{i,0},\ldots,x_{i,n}\rangle .$$

So far, we have the quiver Q, now we want to compute the relations, so we study the natural map

$$kQ \to A$$

given by the universal property of quiver algebras [1, p. II.1.8], and we find a generating set of the kernel. We claim that the kernel is generated by terms

$$R = \{x_{i+1,p}x_{i,q} - x_{i+1,q}x_{i,p} \mid 0 \le p,q \le n, -n \le i \le 0\}.$$

We can see that all of R is in the kernel, since

$$x_p x_q = x_q x_p \in \mathrm{Hom}(\mathcal{O}(i),\mathcal{O}(i+2)).$$

Now, we check that $\langle R\rangle$ is the entire kernel by seeing that $kQ/\langle R\rangle \to A$ is injective. In fact, both k-algebras here are graded by lengths of paths, so we can show the kernel is 0 by showing each graded piece is 0.

Suppose

$$\sum \alpha_{p_0,\ldots,p_r} x_{i+r,p_r}\cdots x_{i,p_0} \in (kQ/\langle R\rangle)_r$$

is in the kernel, i.e.,

$$\sum \alpha_{p_0,\ldots,p_r} x_{p_r} \cdots x_{p_0} = 0 \in \operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(i+r)) = k[x_0, \ldots, x_n]_r,$$

i.e., it is the zero polynomial. This will imply our original term in $kQ/\langle R\rangle$ is 0, since we can use R to put all the monomials in standard order.

In conclusion, we get that A is the algebra for the (bound) **Beilinson $\mathbb{P}^n$ quiver**, i.e., the quiver

$$\begin{tikzcd}
\overset{0}{\bullet} \arrow[r, bend left=50, "x_{1,0}"] \arrow[r, bend left=15, "x_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "x_{1,n}"'] & \overset{1}{\bullet} \arrow[r, bend left=50, "x_{2,0}"] \arrow[r, bend left=15, "x_{2,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "x_{2,n}"'] & \overset{2}{\bullet} & \cdots & \overset{n\text{-}1}{\bullet} \arrow[r, bend left=50, "x_{n,0}"] \arrow[r, bend left=15, "x_{n,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "x_{n,n}"'] & \overset{n}{\bullet}
\end{tikzcd}$$

with the relations

$$x_{p,i+1}x_{q,i} - x_{q,i+1}x_{p,i} = 0.$$

3.2 Main Theorem: LF and RG

Now, we're getting closer to the main theorem, which tells us how nice a tilting sheaf is—in the same way the triangulated category generated by an exceptional object is just $D^b(k)$, we'll see that the triangulated category generated by a tilting sheaf is $D^b(A)$, where A will actually be a bound quiver algebra (though unfortunately we will have to take the opposite quiver with the opposite relations, since we will get a derived equivalence with right modules over A).

First, though, we need a lemma.

Lemma 3.11. Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor, which sends a compact generator $C \in \mathcal{C}$ to a compact generator $F(C) = D$ of $\mathcal{D}$, such that the induced map

$$\operatorname{End}^\bullet_{\mathcal{C}}(C) \to \operatorname{End}^\bullet_{\mathcal{D}}(D)$$

is an isomorphism. Then, F is an equivalence.

Proof. To show at least F is an equivalence, we show it is fully faithful and essentially surjective.

To see fully-faithful, first show that the full subcategory

$$T_C = \{X \in \mathcal{C} \mid \operatorname{Hom}(C, X) \to \operatorname{Hom}(FC, FX) \text{ is a bijection}\}$$

is a strictly full saturated triangulated subcategory containing $C[i]$ for all $i \in \mathbb{Z}$. This is true—it is clearly closed under direct sums, direct summands, and shifts. To see it is closed under cones, notice that for a distinguished triangle $X \to Y \to Z \to X[1]$ we get a commutative diagram of (long) exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Hom}(C, X[1]) \arrow[r] \arrow[d, "F_{C,X[1]}"] & \operatorname{Hom}(C, Z) \arrow[r] \arrow[d, "F_{C,Z}"] & \operatorname{Hom}(C, Y) \arrow[r] \arrow[d, "F_{C,Y}"] & \cdots \\
\cdots \arrow[r] & \operatorname{Hom}(FC, FX[1]) \arrow[r] & \operatorname{Hom}(FC, FZ) \arrow[r] & \operatorname{Hom}(FC, FY) \arrow[r] & \cdots
\end{tikzcd}$$

all vertical morphisms are isomorphisms except for potentially every 3rd morphism, but then we apply the 5 lemma to get $F_{C,Z}$ is also an isomorphism. So, T_C has to be all of $\mathcal{C}$.

The same argument shows that

$$T'_X = \{Y \in \mathcal{C} \mid F_{X,Y} \text{ is bijective}\}$$

is thick (since the previous step told us $C[i] \in T'_X$), so it is also all of $\mathcal{C}$. This establishes that F is fully faithful.

Next, F is essentially surjective, because it is exact and has a generator in its image. Thus, F is an equivalence. □

Theorem 3.12. Let T be a tilting sheaf on a projective smooth X over k, with $A = \mathrm{End}(T)$. Then, the following functors form an equivalence.

$$\begin{aligned} RG = \mathrm{RHom}(T, -) &: D^b(X) \longrightarrow D^b(\text{mod-}A) \\ LF = - \otimes^L_A T &: D^b(\text{mod-}A) \longrightarrow D^b(X) \end{aligned}$$

Proof. First, we more carefully define the functors.

We have the left exact functor $G = \mathrm{Hom}(T, -) : \mathrm{QCoh}(X) \to k\text{-Vec}$, which actually factors through Mod-A because we have an action by precomposition.

The category $\mathrm{QCoh}(X)$ has enough injectives, so we can right derive the functor to get

$$\mathrm{RHom}(T, -) : D^+(\mathrm{QCoh}(X)) \to D^+(\text{Mod-}A),$$

and usual arguments tell us that this descends to

$$RG = \mathrm{RHom}(T, -) : D^b(X) \to D^b(\text{mod-}A),$$

(we have an Ext spectral sequence to ensure that we get bounded complexes because $\mathrm{Coh}(X)$ has finite global dimension, and morphisms between coherent sheaves are finite dimensional to ensure that precomposed with $D^b(X) \cong D^b_{\mathrm{Coh}(X)}(\mathrm{QCoh}(X)) \hookrightarrow D^b(\mathrm{QCoh}(X))$ will with us finitely generated A modules).

In the other direction, we have the functor

$$F = - \otimes_A T : \text{mod-}A \to \mathrm{QCoh}(X),$$

in the following sense.

that given $M \in \text{mod-}A$, we get then $\underline{M}$ is in Mod-$\underline{A}$ and T is in $\underline{A}$-Mod, so we can form the $\underline{A}$-balanced tensor product, which is then inherits an $\mathcal{O}_X$-module structure from T (via left multiplication).

This functor is right exact, and mod-A has enough projectives, so we can left derive it to get

$$- \otimes^L_A T : D^-(\text{mod-}A) \to D^-(\mathrm{QCoh}(X)),$$

and Serre theorems/finite global dimension of A will give sheaf Tor finite dimensions/vanishing to imply that this functor descends to

$$LF = - \otimes_A^L T : D^b(\text{mod-}A) \to D^b(X).$$

Now, we need to show these functors form an equivalence. The assumption T2 for T exactly tells us that $RG(T) = A$, and so we also get a (ring) morphism

$$\text{End}^\bullet(T) \xrightarrow{RF} \text{End}^\bullet(A),$$

which is an isomorphism because in nonzero degrees both sides are 0 (by Ext vanishing or because A is projective), and then in degree 0, it sends an endomorphism $f : T \to T$ to left multiplication by $f \in A$, which are exactly the endomorphisms of A as a right A-module.

So, our lemma tells us that RG is an equivalence, but a-priori we do not know that LF is a quasi-inverse. To see this, we first notice that $F \dashv G$ form an adjunction before we derived anything. To see this (it seems slightly subtle because it varies slightly from usual tensor-hom in the sheafiness of it), we will spell out the unit and counit, and check the triangle identities. We have $\eta : 1_{\text{Mod-}A} \Rightarrow GF$ by

$$\eta_M : M \to \text{Hom}(T, M \otimes_A T)$$

where we send m to the morphism $t \mapsto m \otimes t$ (which first is a map to the tensor presheaf, but then we postcompose to get a map to the sheafification). Then, we have $\varepsilon : FG \Rightarrow 1_{\text{QCoh}(X)}$ by

$$\varepsilon_{\mathcal{E}} : \text{Hom}(T, \mathcal{E}) \otimes_A T \to \mathcal{E}$$

where we send $\varphi \otimes t$ to $\varphi(t)$ (which is a map from the presheaf factoring through the sheafification). Then, one checks that the triangle identities hold by computing the compositions on stalks—where sheafification is not a concern. □

4 Computations

4.1 Computation: LF

Let X be a projective variety with exceptional collection $E_1, \ldots, E_r$, and corresponding tilting sheaf

$$T = \bigoplus_{i=1}^{r} E_i.$$

We'd like to compute this functor

$$LF = (-) \otimes_A^L T : D^b(\text{mod-}A) \to D^b(X)$$

explicitly. Even more, we can concretely understand mod-A as $\text{rep}_k(Q^{\text{op}}, R^{\text{op}})$, and we'll ultimately want to understand the entire composition

$$D^b(\text{rep}_k(Q^{\text{op}}, R^{\text{op}})) \to D^b(X).$$

4.1.1 Setup

To compute LF, as usual, we need to replace either $\underline{M}_\bullet$ or T with a quasi-isomorphic complex of flat $\underline{A}$-modules. To replace $\underline{M}_\bullet$, we make the following observation:

If M is flat in mod-A, then $\underline{M}$ is flat in Mod-$\underline{A}$.

To see this, note that M flat in mod-A implies M is projective because finitely generated modules over Noetherian rings are finitely presented, and any finitely presented flat module is projective. Therefore, M is a direct summand of a free module A^n, so it is still projective in Mod-A, so it is flat in Mod-A.

Now, to check that $\underline{M}$ is flat, we take a short exact sequence in Mod-$\underline{A}$

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

and apply $\underline{M} \otimes_{\underline{A}} -$ to get

$$0 \to \underline{M} \otimes_{\underline{A}} \mathcal{F} \to \underline{M} \otimes_{\underline{A}} \mathcal{G} \to \underline{M} \otimes_{\underline{A}} \mathcal{H} \to 0,$$

which is still exact because taking stalks at $x \in X$ we get

$$0 \to M \otimes_A \mathcal{F}_x \to M \otimes_A \mathcal{G}_x \to M \otimes_A \mathcal{H}_x \to 0,$$

which is the application of $M \otimes_A -$ to the short exact sequence which is the stalks, and we know by assumption M is flat in Mod-A.

So, the moral is that we can replace a given $M_\bullet$ with a quasi-isomorphic complex of flat/projective objects in mod-A to do our computation, which is actually a practical thing to do, whereas it's hard to understand how to resolve T using projective/flat objects in Mod-$\underline{A}$.

4.1.2 Projectives

As in the setup, we should start by understanding how LF maps projective objects. As a first step, we show the following.

Proposition 4.1. $LF(A) \cong A$.

Proof. Since A is projective, $LF(A)$ is computed as $F(A) = A \otimes_A T$. Now, we have an **evaluation map**

$$\begin{aligned} A \otimes_A T &\xrightarrow{\text{eval}} T \\ (A \otimes_A T)(U) &\to T(U) \\ \varphi \otimes t &\mapsto \varphi_U(t) \end{aligned}$$

(which, to be precise, is first defined on the tensor presheaf, and then factors through the tensor sheaf). On stalks, this is the map

$$\begin{aligned} A \otimes_A T_x &\longrightarrow T_x \\ \varphi \otimes t &\longmapsto \varphi_x(t), \end{aligned}$$

and this is an isomorphism, since this is exactly the left action on T_x. In other words, $\varphi \otimes t = 1 \otimes \varphi_x(t)$, so we have an inverse map $t \longmapsto 1_T \otimes t$. $\square$

Next, we want to see how to compute this functor in terms of quiver representations. We saw that

$$A \cong kQ/\langle R\rangle$$

for some quiver Q and relations R, so we get

$$\text{mod-}A \cong kQ^{\text{op}}/\langle R^{\text{op}}\rangle\text{-mod} \cong \text{rep}_k(Q^{\text{op}}, R^{\text{op}}).$$

We have to come to terms with the "opposite" flipping the arrows around, since if we really want to study quiver representations we have to consider left modules (with our conventions).

Recall that a quiver representation is an assignment of vector spaces to vertices and linear maps to arrows satisfying the relations. Now that we've inverted the arrows, we can think of our job as assigning functionals to vertices and pullbacks to arrows. To reinforce this convention, for every arrow $\alpha \in Q$, we will label its opposite arrow $\alpha^* \in Q^{\text{op}}$.

Now, we try to understand how such a representation of Q^{op} is sent over to $D^b(X)$. As in our remark, we should resolve our quiver representation using flat/projective objects, so our task boils down to finding the image of projective quiver representations and the image of morphisms.

Let $1_{E_1}, \ldots, 1_{E_r}$ be the identity maps on each object in our exceptional collection. These are exactly the vertices/length 0 paths in Q, and so $1^*_{E_i}$ are the vertices/length 0 paths in Q^{op}.

The projective objects in $A^{\text{op}} = kQ^{\text{op}}/\langle R^{\text{op}}\rangle$ are then given by

$$P_i = A^{\text{op}} 1_{E_i},$$

which is exactly the k-span of (opposite) paths starting at the vertex 1_{E_i}. In terms of the endomorphism ring, these are the endomorphisms $T \to T$ with image in E_i.

Now, we compute LF on P_i, and since P_i is projective, it is computed just by F. Since P_i is projective, it is a direct summand of A, so the injection $P_i \hookrightarrow A$ is carried to an injection

$$P_i \otimes_A T \hookrightarrow A \otimes_A T,$$

where the right is isomorphic to T by the evaluation map. So, we are really just restricting the evaluation map to the subobject $P_i \otimes_A T$.

Proposition 4.2. *$LF(P_i) \cong E_i$.*

Proof. We compute the image of eval restricted to $P_i \otimes_A T$. First, we see that eval factors through $E_i \leq T$, since

$$\varphi \otimes t \mapsto \varphi_U(t) \in E_i$$

when φ has image in E_i. So, we get a well-defined

$$P_i \otimes_A T \xrightarrow{\text{eval}} E_i.$$

This is at least an injection, since it is a restriction (of the codomain) of the composition of two injections.

So, to see that this is an isomorphism, we just show that the stalk maps are surjections. The stalk maps are given by

$$\begin{aligned} P_i \otimes_A T_x &\longrightarrow (E_i)_x \\ \varphi \otimes t &\longmapsto \varphi_x(t), \end{aligned}$$

and indeed for $t \in (E_i)_x$ if we take $\pi_i : T \to E_i$ to be projection onto E_i, then we have preimage $\pi_i \otimes t$. $\square$

Finally, to aid in our computation, we should describe how F sends morphisms $P_i \xrightarrow{f^*} P_j$ to morphisms $E_i \to E_j$, in the sense of making the diagram commute.

$$\begin{tikzcd} P_i \otimes_A T \arrow[r, "f^* \otimes 1_T"] \arrow[d] & P_j \otimes_A T \arrow[d] \\ E_i \arrow[r] & E_j \end{tikzcd}$$

Note that we continue making the convention of thinking of elements of A^{op} modules as functionals, so we denote the morphism $P_i \to P_j$ by f^* (this is maybe a bad convention (as with all my other conventions)). In any case, such a morphism is determined by its restriction

$$1^*_{E_i} P_i \longrightarrow 1^*_{E_i} P_j,$$

which in terms of endomorphisms of T, is

$$k \cdot \{\pi_i\} \longrightarrow k \cdot \{\text{morphisms } E_i \to E_j\},$$

just a choice of endomorphism $f^*(\pi_i)$ (where π_i is really the same as 1_{E_i}).

Indeed, if we know this restriction, we get from our commutative diagram for a section $s \in E_i$

$$\begin{tikzcd} \pi_i \otimes s \arrow[r, mapsto] & f^*(\pi_i) \otimes s \arrow[d, mapsto] \\ s \arrow[u, mapsto] \arrow[r, mapsto] & f^*(\pi_i)(s) \end{tikzcd}$$

In summary, $P_i \xrightarrow{f^*} P_j$ is sent to $f^*(1^*_{E_i}) \in 1^*_{E_i} P_j = \mathrm{Hom}(E_i, E_j)$.

4.1.3 Examples

Consider

- $X = \mathbb{P}^1$,
- exceptional collection $\mathcal{O}(-1), \mathcal{O}$
- tilting sheaf $T = \mathcal{O}(-1) \oplus \mathcal{O}$

- endomorphism algebra

$$A = \mathrm{End}_{\mathcal{O}_{\mathbb{P}^1}}(T) = \begin{bmatrix} \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(-1)) & \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}) \\ 0 & \mathrm{Hom}(\mathcal{O}, \mathcal{O}) \end{bmatrix}$$

corresponding to quiver

$$Q = \left[\begin{tikzcd} \overset{1_{\mathcal{O}(-1)}}{\bullet} \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & \overset{1_{\mathcal{O}}}{\bullet} \end{tikzcd} \right]$$

and opposite quiver

$$Q^{\mathrm{op}} = \left[\begin{tikzcd} \overset{1^*_{\mathcal{O}(-1)}}{\bullet} & \overset{1^*_{\mathcal{O}}}{\bullet} \arrow[l, bend right, "x^*"'] \arrow[l, bend left, "y^*"] \end{tikzcd} \right]$$

Now, let

$$M = \left[\begin{tikzcd} \overset{V}{\bullet} & \overset{W}{\bullet} \arrow[l, bend right, "S^*"'] \arrow[l, bend left, "T^*"] \end{tikzcd} \right]$$

be a representation of Q^{op}. Then, this has a projective resolution as in Construction 2.9 by (labeling the direct summands on the left term)

$$0 \to (W \otimes_k P_{-1}) \oplus (W \otimes_k P_{-1}) \to V \otimes_k P_{-1} \oplus W \otimes_k P_0 \to M \to 0,$$

where the first map is determined by the restriction to the k-vector space

$$W^{\oplus 2} \cong (W \otimes_k 1^*_{\mathcal{O}(-1)} P_{-1})^{\oplus 2} \hookrightarrow (W \otimes_k P_{-1})^{\oplus 2},$$

which is given by the matrix

$$\begin{array}{cc} & \begin{array}{cc} W & W \end{array} \\ \begin{array}{c} V \otimes_k P_{-1} \\ W \otimes_k P_0 \end{array} & \begin{bmatrix} S^*(-) \otimes 1^*_{\mathcal{O}(-1)} & T^*(-) \otimes 1^*_{\mathcal{O}(-1)} \\ (-) \otimes x^* & (-) \otimes y^* \end{bmatrix} \end{array}$$

Now, truncating M, we get a quasi-isomorphic projective complex, which is sent to

$$0 \to W \otimes_k \mathcal{O}(-1) \oplus W \otimes_k \mathcal{O}(-1) \to V \otimes_k \mathcal{O}(-1) \oplus W \otimes_k \mathcal{O} \to 0,$$

where the map is given by the following matrix

$$\begin{array}{cc} & \begin{array}{cc} W \otimes_k \mathcal{O}(-1) & W \otimes_k \mathcal{O}(-1) \end{array} \\ \begin{array}{c} V \otimes_k \mathcal{O}(-1) \\ W \otimes_k \mathcal{O} \end{array} & \begin{bmatrix} S^* \otimes 1 & T^* \otimes 1 \\ 1_W \otimes x & 1_W \otimes y \end{bmatrix} \end{array}$$

which can be seen by fixing an argument $w \in W$ and studying compositions like

$$P_{-1} \xrightarrow{\cong} w \otimes_k P_{-1} \to S^*(w) \otimes_k P_{-1} \xrightarrow{\cong} P_{-1}$$

where we know how to compute the entire composition by the discussion in the setup.

4.1.4 Simples

The equivalence $LF : D^b(\text{mod-}A) \to D^b(\mathbb{P}^n)$ endows $D^b(\mathbb{P}^n)$ with a heart of a t-structure by the image

$$\text{mod-}A \hookrightarrow D^b(\text{mod-}A) \to D^b(\mathbb{P}^n).$$

To try to characterize this heart, we'll find all the simples of mod-A, compute their images in $D^b(\mathbb{P}^n)$, and then our heart will be the extension closure of those simples.

This is done more efficiently in the next section, the duality of simples and projectives. Nonetheless we sketch the computation for projective space.

The simple quiver representations are those assigning k to a single vertex, and 0 to the remaining vertices (and arrows). To compute the image of this representation under LF, we need to write a projective resolution of the simple representation, which we claim looks like a twisted truncated from the left Koszul complex. Roughly this is because to surject onto the simple S_i concentrated at $\mathcal{O}(i)$, we will use one copy of $P_i = A^{\mathrm{op}} 1_{\mathcal{O}(i)}$, and this surjection has a kernel K generated by $1_{\mathcal{O}(i-1)}K$ which is dimension $n+1$ (for the $n+1$ arrows from $\mathcal{O}(i-1) \to \mathcal{O}(i)$), and then we will surject onto the kernel using P_{i-1}, so our resolution looks like

$$\cdots \to P_{i-1} \otimes k^{n+1} \to P_i \to S_i \to 0,$$

and the kernel K of $P_{i-1} \otimes k^{n+1} \to P_i$ will be generated by $1_{\mathcal{O}(i-2)}K$, which has dimension

$$(n+1)^2 - (\binom{n+1}{2} + n + 1) = \binom{n+1}{2} = \dim \bigwedge^2 k^{n+1},$$

and so on.

Now, applying LF to these projective resolutions, we will get truncated Koszul complexes, which only have cohomology in the leftmost degree, which is just the kernel. So, we need the following proposition.

Proposition 4.3. Let $V = k^{n+1}$, and

$$0 \to \bigwedge^{n+1} V \otimes_k \mathcal{O}(-n-1) \xrightarrow{d_{n+1}} \cdots \to \bigwedge^2 V \otimes_k \mathcal{O}(-2) \xrightarrow{d_2} \bigwedge^1 V \otimes_k \mathcal{O}(-1) \xrightarrow{d_1} \bigwedge^0 V \otimes_k \mathcal{O} \xrightarrow{d_0} 0$$

the Koszul complex. Then, $\ker d_i = \Omega^i$.

Proof. Clearly $\ker d_0 = \mathcal{O} = \Omega^0$, and the Euler sequence tells us $\ker d_1 = \Omega^1$.

In general, we will still analyze the Euler sequence

$$0 \to \Omega^1 \to V \otimes_k \mathcal{O}(-1) \to \mathcal{O} \to 0.$$

In general, it is hard to say how exterior powers of vector bundles relate to each other in a short exact sequence

$$0 \to L \to M \to N \to 0.$$

We can, however, at least say that $\bigwedge^p M$ has a filtration

$$\bigwedge^p M = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^{p+1} = 0,$$

where $F^i/F^{i+1} \cong \bigwedge^i L \otimes \bigwedge^{p-i} N$ by setting F^i to be the subbundle of M where at least i wedges come from L.

In our situation, all the quotients F^i/F^{i+1} will be trivial except for $i = p-1, p$ because $N = \mathcal{O}$ is a line bundle. The remaining interesting bit of the filtration is

$$\bigwedge^p V \otimes_k \mathcal{O}(-p) = F^{p-1} \supseteq F^p \supseteq 0,$$

which gives us a short exact sequence

$$0 \to \Omega^p \to \otimes \bigwedge^p V \otimes_k \mathcal{O}(-1) \to \Omega^{p-1} \to 0,$$

and indeed one can check that the composition (where we inductively assume Ω^{p-1} is the kernel of the next differential)

$$\bigwedge^p V \otimes_k \mathcal{O}(-p) \twoheadrightarrow \Omega^{p-1} \hookrightarrow \bigwedge^{p-1} V \otimes_k \mathcal{O}(-p+1)$$

is the map in the Koszul complex. Thus, we have computed the kernel. $\square$

4.2 Computation: RG on Projective Space

The functor

$$RG = \mathrm{RHom}(T, -) : D^b(X) \to D^b(\text{mod-}A)$$

is harder to compute on the nose—but we can settle for computing the cohomology of $RG(\mathcal{E}^\bullet)$.

In the case $X = \mathbb{P}^n$ with the tilting sheaf $T = \bigoplus_{i=-n}^0 \mathcal{O}(i)$, and considering

$$\mathrm{Coh}(\mathbb{P}^n) \hookrightarrow D^b(\mathbb{P}^n) \qquad \text{mod-}A \hookrightarrow D^b(\text{mod-}A)$$

as degree 0 complexes, we can characterize $RG(\mathrm{Coh}(\mathbb{P}^n)) \cap \text{mod-}A$ as follows.

Proposition 4.4. A sheaf $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^n)$ has $RG(\mathcal{F}) \in \text{mod-}A$ a single quiver representation (i.e., a degree 0 complex) exactly when

$$H^j(\mathcal{F}(i)) = 0$$

for $i = 0, \ldots, n$ and $j > 0$.

Proof. To check $RG(\mathcal{F}) \in \text{mod-}A$, we need to show

$$H^j(RG(\mathcal{F})) = 0$$

for $j \neq 0$. Now, to compute RG, we have a have the following commutative diagram (composition of derived functors because $\mathcal{H}om(T, -)$ sends injective sheaves to flasque sheaves).

$$\begin{tikzcd} D^b(\mathrm{QCoh}(\mathbb{P}^n)) \arrow[r, "R\mathcal{H}om(T,-)"] \arrow[rr, bend right, "R\mathrm{Hom}(T,-)"'] & D^b(\text{Mod-}\mathcal{E}nd(T)) \arrow[r, "R\Gamma"] & D^b(\text{Mod-}A) \end{tikzcd}$$

So, to compute $RG(\mathcal{F})$, we first compute

$$R\mathcal{H}om(T, \mathcal{F}) = \mathcal{H}om(T, \mathcal{F}),$$

since T is locally free. Then, we can compute

$$\begin{aligned} R^j\Gamma(\mathcal{H}om(T, \mathcal{F})) &= H^j(\mathcal{H}om(\bigoplus_{i=-n}^{0} \mathcal{O}(i), \mathcal{F})) \\ &= H^j(\bigoplus_{i=0}^{n} \mathcal{F}(i)) \\ &= \bigoplus_{i=0}^{n} H^j(\mathcal{F}(i)). \end{aligned}$$

So, $RG(\mathcal{F}) \in \text{mod-}A$ exactly when each summand vanishes for $j > 0$, which is statement of the proposition. □

Remark 4.5. We can get this vanishing via an assumption on the **Castelnuovo-Mumford regularity** of $\mathcal{F}$. In particular, when $\mathcal{F}$ is 1-regular, we get

$$H^j(\mathcal{F}(1 + p - j)) = 0, \qquad p \geq 0,$$

so by taking $p = i + j$ for $i = 0, \ldots, n$ and $j > 0$, we get all the desired cohomology vanishing.

Somehow, regularity is not quite the right condition here—we need a sort of "rectangular" vanishing to be in the heart, whereas regularity is "diagonal".

I believe we can get regularity to be more pertinent if we study a *dual* exceptional collection and its corresponding tilt.

5 Duality of Simples and Projectives

As we saw previously, the *projective objects* of the heart of the t-structure from the tilting equivalence (from a full strong exceptional collection) correspond to the exceptional objects themselves.

It is natural to ask if there is a nice description of the *simple objects* of the heart. It turns out that the simple objects correspond to exceptional objects of a *dual* exceptional collection.

The first thing is to establish that for an exceptional object $E \in \mathcal{D}$, the inclusion $\langle E \rangle \cong D^b(\mathbb{C}) \hookrightarrow \mathcal{D}$ is *admissible.*

5.1 Admissibility and Mutations

Definition 5.1. A triangulated subcategory $\mathcal{A} \subseteq \mathcal{D}$ is **left (resp. right) admissible** if the inclusion

$$i : \mathcal{A} \hookrightarrow \mathcal{D}$$

admits a left (resp. right) adjoint i^* (resp. $i^!$).

The adjoint should be thought of as a sort of projection onto $\mathcal{A}$, and the cone over the unit/counit to be the orthogonal complement.

For example, suppose $i : \mathcal{A} \hookrightarrow \mathcal{D}$ is left admissible, so that there is a left adjoint $i^* : \mathcal{D} \to \mathcal{A}$. Then for an object $X \in \mathcal{D}$, we have the unit map

$$X \to ii^*X$$

comparing X to the projection of X onto $\mathcal{A}$, which we can extend to a distinguished triangle

$$B \to X \to ii^*X \to B[1].$$

Now, observe that for any $A \in \mathcal{A}$, after applying $\mathrm{Hom}(-, iA)$ to get a LES, we get

$$\mathrm{Hom}_{\mathcal{D}}(ii^*X, iA) = \mathrm{Hom}_{\mathcal{A}}(i^*X, A) = \mathrm{Hom}_{\mathcal{D}}(X, iA),$$

so by the LES in $\mathrm{Hom}_{\mathcal{D}}(-, iA)$, we get $\mathrm{Hom}_{\mathcal{D}}(B, iA) = 0$, i.e., $B \in {}^{\perp}\mathcal{A}$.

Definition 5.2. Let $X \in \mathcal{D}$, $i : \mathcal{A} \to \mathcal{D}$ the inclusion, and $i^*, i^!$ the left/right adjoints (if they exist). Denote

$$L_{\mathcal{A}}(X) := C(ii^!X \to X) \in \mathcal{A}^{\perp}.$$
$$R_{\mathcal{A}}(X) := C(X \to ii^*X)[-1] \in {}^{\perp}\mathcal{A}$$

which are called the **left/right mutations of X through $\mathcal{A}$**.

Maybe more intuitive notation, keeping in mind the inner product space analogies, would be something like this.

$$\mathrm{proj}^{\perp}_{\mathcal{A}} X := i^*X, \qquad \mathrm{orth}^{\perp}_{\mathcal{A}} X := L_{\mathcal{A}}(X),$$
$${}^{\perp}\mathrm{proj}_{\mathcal{A}} X := i^!X, \qquad {}^{\perp}\mathrm{orth}_{\mathcal{A}} X := R_{\mathcal{A}}(X).$$

So far $i^*, i^!$, if they exist, are functorial (either by definition, or if we just define $i^*, i^!$ objectwise by satisfying a universal property from the adjunction, we also automatically get functoriality).

However, it is not immediately clear if $L_{\mathcal{A}}, R_{\mathcal{A}}$ are functorial—they are at least well-defined up to isomorphism because cones are well-defined up to isomorphism, but famously cones in general lack functoriality. So, we should hope something special is going on here.

Proposition 5.3. Mutations assemble into functors $L_{\mathcal{A}} : \mathcal{D} \to {}^{\perp}\mathcal{A}$ and $R_{\mathcal{A}} : \mathcal{D} \to \mathcal{A}^{\perp}$.

Proof. We just show $R_{\mathcal{A}}$ is functorial. Let $f : X \to Y$ be a morphism in $\mathcal{D}$, so we get the following commutative diagram by the functoriality of $i^!$ and TR3.

$$\begin{tikzcd}
i^! X \arrow[r] \arrow[d, "i^* f"] & X \arrow[r] \arrow[d, "f"] & R_{\mathcal{A}}(X) \arrow[r] \arrow[d, dashed] & i^* X[1] \arrow[d, "i^! f[1]"] \\
i^! Y \arrow[r] & Y \arrow[r] & R_{\mathcal{A}}(Y) \arrow[r] & i^! Y[1]
\end{tikzcd}$$

However, we do not know if there is a unique map making this diagram commute. By subtracting, we can assume $f = 0$, $i^! f = 0$, so our task is to show that

$$R_{\mathcal{A}}(X) \xrightarrow{0} R_{\mathcal{A}}(Y)$$

is the unique map making the diagram commute. By Yoneda, we only need to show

$$\mathrm{Hom}_{{}^{\perp}\mathcal{A}}(B, R_{\mathcal{A}}(X)) \to \mathrm{Hom}_{{}^{\perp}\mathcal{A}}(B, R_{\mathcal{A}}(Y))$$

is 0 for all $B \in {}^{\perp}\mathcal{A}$. Hitting the morphism of distinguished triangles with $\mathrm{Hom}(B, -)$, we get

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(B, X) \arrow[r] \arrow[d] & \mathrm{Hom}(B, R_{\mathcal{A}}(X)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(B, Y) \arrow[r] & \mathrm{Hom}(B, R_{\mathcal{A}}(Y)) \arrow[r] & 0
\end{tikzcd}$$

and so $\mathrm{Hom}(B, X) \xrightarrow{0} \mathrm{Hom}(B, Y)$ forces our desired morphism to be 0.

A similar argument then shows that $L_{\mathcal{A}}$ is functorial. □

One other key result about mutations.

Proposition 5.4. The restrictions $L_{\mathcal{A}} : {}^{\perp}\mathcal{A} \to \mathcal{A}^{\perp}$ and $R_{\mathcal{A}} : \mathcal{A}^{\perp} \to {}^{\perp}\mathcal{A}$ form an equivalence of categories.

Proof. write down a proof. □

Example 5.5. Let S be a smooth scheme over k and E a vector bundle. Consider the projective bundle $\pi : \mathbb{P}(E) \to S$. Then,

$$\pi^* : D^b(S) \hookrightarrow D^b(\mathbb{P}(E))$$

is full faithful, and this is a right admissible subcategory (it will also be left admissible by Grothendieck duality).

To see this, first we check that π^* is fully faithful, we should check

$$\mathrm{Hom}(E, F) \xrightarrow{\pi^*} \mathrm{Hom}(\pi^* E, \pi^* F)$$

is an isomorphism, and to check this we refer to the triangle

$$\begin{tikzcd} \mathrm{Hom}(E,F) \arrow[r] \arrow[dr] & \mathrm{Hom}(\pi^*E,\pi^*F) \arrow[d] \\ & \mathrm{Hom}(E, R\pi_*\pi^*F) \end{tikzcd}$$

where the vertical map comes from the $\pi^* \vdash R\pi_*$ adjunction, and the diagonal map is post-composition with the unit of the adjunction. So, to see the horizontal map is an isomorphism, we just check that the diagonal map is an isomorphism, so we need the unit to be an isomorphism.

To see that $E \to R\pi_*\pi^*E$, it suffices to prove

$$\mathcal{O}_S \to R\pi_*\mathcal{O}_{\mathbb{P}(E)}$$

is an isomorphism, since we can just tensor it with E and apply the projection formula.

We work locally. Over an affine open $\operatorname{Spec} A$ that trivializes $\mathbb{P}(E)$, the map π is the structure map

$$\pi : \mathbb{P}^r_A \to \operatorname{Spec} A$$

of projective space, and $R\pi_*$ is $R\Gamma$, i.e., sheaf cohomology (up to tilde). Now, this is something well-known:

$$R\Gamma\mathcal{O}_{\mathbb{P}(E)}|_{\operatorname{Spec} A} = A,$$

which is $\mathcal{O}_S|_{\operatorname{Spec} A}$ (up to tilde).

So, π^* is fully faithful, and now it is right admissible because π^* has adjoint $R\pi_*$.

5.2 Exceptional Implies Admissible

Proposition 5.6. For an exceptional object $E \in \mathcal{D}$, the inclusion

$$\langle E\rangle = D^b(\mathrm{pt}) \hookrightarrow \mathcal{D}$$

is left and right admissible.

Proof. Let $X \in \mathcal{D}$, we define

$$i^!X := \mathrm{Hom}^\bullet_{\mathcal{D}}(E,X) \otimes E.$$

(This seems like a reasonable candidate for projecting X onto $\langle E\rangle$, and it's functorial). To see that this is a right adjoint to $i : \langle E\rangle \hookrightarrow \mathcal{D}$, we have to produce a natural isomorphism

$$\mathrm{Hom}(E, i^!X) \cong \mathrm{Hom}(iE, X)$$

for $X \in \mathcal{D}$. A priori, the right argument should be any object in $\langle E\rangle$, but these are all just direct sums $\bigoplus_j E^{a_j}[j]$.

Now, this natural isomorphism follows from the definition of $i^!X$, as

$$\mathrm{Hom}(E, \mathrm{Hom}^\bullet(E,X) \otimes E) \cong \mathrm{Hom}(E,E) \otimes \mathrm{Hom}^\bullet(E,X) \cong \mathrm{Hom}^\bullet(E,X).$$

Explicitly, this isomorphism works by interpreting a

$$\varphi : E \to \mathrm{Hom}^\bullet_{\mathcal{D}}(E,X) \otimes E,$$

as a column vector valued in $\mathrm{Hom}(E,E) = k1_E$ indexed by a basis of $\mathrm{Hom}^\bullet_{\mathcal{D}}(E,X)$, so a formal sum

$$\varphi = \sum_{\psi \in \mathrm{Hom}^\bullet(E,X)} c_\psi 1_E \otimes \psi,$$

which is then mapped to evaluation of the formal sum

$$\sum_{\psi \in \mathrm{Hom}^\bullet(E,X)} c_\psi \psi.$$

So, in this way, we see $\langle E \rangle$ is right admissible, and similarly we can define

$$i^*X := \mathrm{Hom}^\bullet_{\mathcal{D}}(X,E)^* \otimes E$$

to see $\langle E \rangle$ is left admissible. $\square$

Now, let's understand how to mutate through an exceptional object, so we need to understand the unit/counit for these adjunctions.

To study the left mutation, we look at the right adjoint

$$i^! = \mathrm{Hom}^\bullet(E, -) \otimes E.$$

The counit is given by the image of $1_{i^!X}$

$$\mathrm{Hom}(i^!X, i^!X) \xrightarrow{\cong} \mathrm{Hom}(ii^!X, X).$$

If we consider

$$1_{i^!X} \in \mathrm{Hom}(i^!X, i^!X) = \mathrm{Hom}(\mathrm{Hom}^\bullet(E,X) \otimes E, \mathrm{Hom}^\bullet(E,X) \otimes E)$$

as the identity matrix, in the sense of a matrix valued in $\mathrm{Hom}(E,E)$ with rows and columns indexed by a basis $\mathrm{Hom}^\bullet(E,X)$, then $1_{i^!X}$ is sent to "summing the columns", which is nothing other than evaluation

$$\mathrm{Hom}^\bullet(E,X) \otimes E \longrightarrow X.$$

So, $L_E(X)$ is defined by the distinguished triangle

$$\mathrm{Hom}^\bullet(E,X) \otimes E \xrightarrow{\text{eval}} X \to L_E(X).$$

To study the right mutation, we look at the left adjoint

$$i^* = \mathrm{Hom}^\bullet(-,E)^* \otimes E.$$

The unit is then given by the image of $1_{1^* X}$

$$\mathrm{Hom}(i^* X, i^* X) \xrightarrow{\cong} \mathrm{Hom}(X, ii^* X).$$

The identity morphism on $\mathrm{Hom}^\bullet(X, E)^* \otimes E$, after picking a basis

$$\mathrm{Hom}^\bullet(X, E) = \langle \psi_1, \ldots, \psi_n \rangle$$

is the identity morphism on each $\psi_i^* \otimes E$ component. Then, the adjoint morphism

$$X \xrightarrow{\text{coeval}} \mathrm{Hom}^\bullet(X, E)^* \otimes E$$

is defined on each component by the map $X \to \psi_i^* \otimes E$ given by ψ_i. Interpreted coordinate free, this is "coevaluation", which is the curried evaluation map

$$X \otimes \mathrm{Hom}^\bullet(X, E) \to E.$$

5.3 Dual Exceptional Collections

TODO.

References

[1] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the Representation Theory of Associative Algebras*. Cambridge University Press, 2006.

[2] The Stacks Project Authors. *Stacks Project*. URL: `https://stacks.math.columbia.edu/`.

[3] Winfried Bruns and H. Jürgen Herzog. *Cohen-Macaulay Rings*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[4] John R. Calabrese. *On a Theorem by Beilinson*. Available at `http://poisson.phc.unipi.it/~calabrese/`. 2009.

[5] Alastair Craw. "EXPLICIT METHODS FOR DERIVED CATEGORIES OF SHEAVES". In: 2007. URL: `https://api.semanticscholar.org/CorpusID:19291767`.